

A DEVELOPMENT OF
THE EARTH'S GRAVITATIONAL POTENTIAL
FROM THE SIXTH THROUGH
THE TWENTY-FIFTH ZONAL HARMONIC

by

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INTRODUCTION

The purpose of this report is to express that part of the earth's gravitational potential from the sixth zonal harmonic through the twenty-fifth, along with its partial derivatives with respect to an inertial geocentric coordinate system, in a form suitable for inclusion in the numerical integration of position partial derivatives for an earth satellite.

EQUATIONS OF MOTION

The earth's potential is

$$U = \frac{\mu}{r} \left[1 - \sum_{n=2}^{\infty} \left(\frac{R}{r} \right)^n J_n P_n (\sin \phi) \right],$$

where

$$\mu = GM$$

$$R = \text{radius of the earth}$$

$$J_n = \text{zonal harmonic coefficients } (n = 2, 3, \dots)$$

$$P_n = \text{Legendre polynomials } (n = 2, 3, \dots)$$

$$\phi = \text{geocentric latitude.}$$

The Legendre polynomials are

$$P_6 (\sin \phi) = \frac{1}{16} (231 \sin^6 \phi - 315 \sin^4 \phi + 105 \sin^2 \phi - 5)$$

$$P_7(\sin \phi) = \frac{1}{16} (429 \sin^7 \phi - 693 \sin^5 \phi + 315 \sin^3 \phi - 35 \sin \phi)$$

$$P_8(\sin \phi) = \frac{1}{128} (6435 \sin^8 \phi - 12012 \sin^6 \phi + 6930 \sin^4 \phi - 1260 \sin^2 \phi + 35)$$

$$P_9(\sin \phi) = \frac{1}{128} (12155 \sin^9 \phi - 25740 \sin^7 \phi + 18018 \sin^5 \phi - 4620 \sin^3 \phi + 315 \sin \phi)$$

$$P_{10}(\sin \phi) = \frac{1}{256} (46189 \sin^{10} \phi - 109395 \sin^8 \phi + 90090 \sin^6 \phi - 30030 \sin^4 \phi$$

$$+ 3465 \sin^2 \phi - 63)$$

$$P_{11}(\sin \phi) = \frac{1}{256} (88179 \sin^{11} \phi - 230945 \sin^9 \phi + 218790 \sin^7 \phi - 90090 \sin^5 \phi$$

$$+ 15015 \sin^3 \phi - 693 \sin \phi)$$

$$P_{12}(\sin \phi) = \frac{1}{1024} (676039 \sin^{12} \phi - 1939938 \sin^{10} \phi + 2078505 \sin^8 \phi - 1021020 \sin^6 \phi$$

$$+ 225225 \sin^4 \phi - 18018 \sin^2 \phi + 231)$$

$$P_{13}(\sin \phi) = \frac{1}{1024} (1300075 \sin^{13} \phi - 4056234 \sin^{11} \phi + 4849845 \sin^9 \phi - 2771340 \sin^7 \phi \\ + 765765 \sin^5 \phi - 90090 \sin^3 \phi + 3003 \sin \phi)$$

$$P_{14}(\sin \phi) = \frac{1}{2048} (5014575 \sin^{14} \phi - 16900975 \sin^{12} \phi + 22309287 \sin^{10} \phi - 14549535 \sin^8 \phi \\ + 4849845 \sin^6 \phi - 765765 \sin^4 \phi + 45045 \sin^2 \phi - 429)$$

$$P_{15}(\sin \phi) = \frac{1}{2048} (9694845 \sin^{15} \phi - 35102025 \sin^{13} \phi + 50702925 \sin^{11} \phi - 37182145 \sin^9 \phi \\ + 14549535 \sin^7 \phi - 2909907 \sin^5 \phi + 255255 \sin^3 \phi - 6435 \sin \phi)$$

$$P_{16}(\sin \phi) = \frac{1}{32768} (300540195 \sin^{16} \phi - 1163381400 \sin^{14} \phi + 1825305300 \sin^{12} \phi \\ - 1487285800 \sin^{10} \phi + 669278610 \sin^8 \phi - 162954792 \sin^6 \phi \\ + 19399380 \sin^4 \phi - 875160 \sin^2 \phi + 6435)$$

$$P_{17}(\sin \phi) = \frac{1}{32768} (583401555 \sin^{17} \phi - 2404321560 \sin^{15} \phi + 4071834900 \sin^{13} \phi \\ - 3650610600 \sin^{11} \phi + 1859107250 \sin^9 \phi - 535422888 \sin^7 \phi \\ + 81477396 \sin^5 \phi - 5542680 \sin^3 \phi + 109395 \sin \phi)$$

$$P_{18}(\sin \phi) = \frac{1}{65536} (2268783825 \sin^{18} \phi - 9917826435 \sin^{16} \phi + 18032411700 \sin^{14} \phi \\ - 17644617900 \sin^{12} \phi + 10039179150 \sin^{10} \phi - 3346393050 \sin^8 \phi \\ + 624660036 \sin^6 \phi - 58198140 \sin^4 \phi + 2078505 \sin^2 \phi - 12155)$$

$$\begin{aligned}
P_{19}(\sin \phi) = & \frac{1}{65536} (4418157975 \sin^{19} \phi - 20419054425 \sin^{17} \phi + 39671305740 \sin^{15} \phi \\
& - 42075627300 \sin^{13} \phi + 26466926850 \sin^{11} \phi - 10039179150 \sin^9 \phi \\
& + 2230928700 \sin^7 \phi - 267711444 \sin^5 \phi + 14549535 \sin^3 \phi - 230945 \sin \phi)
\end{aligned}$$

$$\begin{aligned}
P_{20}(\sin \phi) = & \frac{1}{262144} (34461632205 \sin^{20} \phi - 167890003050 \sin^{18} \phi + 347123925225 \sin^{16} \phi \\
& - 396713057400 \sin^{14} \phi + 273491577450 \sin^{12} \phi - 116454478140 \sin^{10} \phi \\
& + 30117537450 \sin^8 \phi - 4461857400 \sin^6 \phi + 334639305 \sin^4 \phi \\
& - 9699690 \sin^2 \phi + 46189)
\end{aligned}$$

$$\begin{aligned}
P_{21}(\sin \phi) = & \frac{1}{262144} (67282234305 \sin^{21} \phi - 344616322050 \sin^{19} \phi + 755505013725 \sin^{17} \phi \\
& - 925663800600 \sin^{15} \phi + 694247850450 \sin^{13} \phi - 328189892940 \sin^{11} \phi \\
& + 97045398450 \sin^9 \phi - 17210021400 \sin^7 \phi + 1673196525 \sin^5 \phi \\
& - 74364290 \sin^3 \phi + 969969 \sin \phi)
\end{aligned}$$

$$\begin{aligned}
P_{22}(\sin \phi) = & \frac{1}{524288} (263012370465 \sin^{22} \phi - 1412926920405 \sin^{20} \phi + 3273855059475 \sin^{18} \phi \\
& - 4281195077775 \sin^{16} \phi + 3471239252250 \sin^{14} \phi - 1805044411170 \sin^{12} \phi \\
& + 601681470390 \sin^{10} \phi - 124772655150 \sin^8 \phi + 15058768725 \sin^6 \phi \\
& - 929553625 \sin^4 \phi + 22309287 \sin^2 \phi - 88179)
\end{aligned}$$

$$\begin{aligned}
P_{23}(\sin \phi) = & \frac{1}{524288} (514589420475 \sin^{23} \phi - 2893136075115 \sin^{21} \phi + 7064634602025 \sin^{19} \phi \\
& - 9821565178425 \sin^{17} \phi + 8562390155550 \sin^{15} \phi - 4859734953150 \sin^{13} \phi \\
& + 1805044411170 \sin^{11} \phi - 429772478850 \sin^9 \phi + 62386327575 \sin^7 \phi \\
& - 5019589575 \sin^5 \phi + 185910725 \sin^3 \phi - 2028117 \sin \phi)
\end{aligned}$$

$$\begin{aligned}
P_{24}(\sin \phi) = & \frac{1}{4194304} (8061900920775 \sin^{24} \phi - 47342226683700 \sin^{22} \phi + 121511715154830 \sin^{20} \phi \\
& - 178970743251300 \sin^{18} \phi + 166966608033225 \sin^{16} \phi - 102748681866600 \sin^{14} \phi \\
& + 42117702927300 \sin^{12} \phi - 11345993441640 \sin^{10} \phi + 1933976154825 \sin^8 \phi \\
& - 194090796900 \sin^6 \phi + 10039179150 \sin^4 \phi - 202811700 \sin^2 \phi + 676039)
\end{aligned}$$

$$\begin{aligned}
P_{25}(\sin \phi) = & \frac{1}{4194304} (15801325804719 \sin^{25} \phi - 96742811049300 \sin^{23} \phi + 260382246760350 \sin^{21} \phi \\
& - 405039050516100 \sin^{19} \phi + 402684172315425 \sin^{17} \phi - 267146572853160 \sin^{15} \phi \\
& + 119873462177700 \sin^{13} \phi - 36100888223400 \sin^{11} \phi + 7091245901025 \sin^9 \phi \\
& - 859544957700 \sin^7 \phi + 58227239070 \sin^5 \phi - 1825305300 \sin^3 \phi \\
& + 16900975 \sin \phi).
\end{aligned}$$

We consider an orthogonal, earth-centered inertial coordinate system in which the z-axis coincides with the earth's axis of rotation. Then,

$$z = r \sin \phi$$

$$r = \sqrt{x^2 + y^2 + z^2}.$$

For the disturbing function

$$F = \frac{\mu}{r} \sum_{n=6}^{25} \left(\frac{R}{r} \right)^n J_n P_n (\sin \phi),$$

we have

$$\begin{aligned} F = \frac{\mu}{r} & \left[\frac{J_6 R^6}{16 r^{12}} (231 z^6 - 315 z^4 r^2 + 105 z^2 r^4 - 5 r^6) + \frac{J_7 R^7 z}{16 r^{14}} (429 z^6 - 693 z^4 r^2 \right. \\ & \quad \left. + 315 z^2 r^4 - 35 r^6) \right. \\ & + \frac{J_8 R^8}{128 r^{16}} (6435 z^8 - 12012 z^6 r^2 + 6930 z^4 r^4 - 1260 z^2 r^6 + 35 r^8) \\ & + \frac{J_9 R^9 z}{128 r^{18}} (12155 z^8 - 25740 z^6 r^2 + 18018 z^4 r^4 - 4620 z^2 r^6 + 315 r^8) \\ & + \frac{J_{10} R^{10}}{256 r^{20}} (46189 z^{10} - 109395 z^8 r^2 + 90090 z^6 r^4 - 30030 z^4 r^6 + 3465 z^2 r^8 - 63 r^{10}) \\ & + \frac{J_{11} R^{11} z}{256 r^{22}} (88179 z^{10} - 230945 z^8 r^2 + 218790 z^6 r^4 - 90090 z^4 r^6 + 15015 z^2 r^8 - 693 r^{10}) \\ & + \frac{J_{12} R^{12}}{1024 r^{24}} (676039 z^{12} - 1939938 z^{10} r^2 + 2078505 z^8 r^4 - 1021020 z^6 r^6 + 225225 z^4 r^8 \\ & \quad - 18018 z^2 r^{10} + 231 r^{12}) \\ & + \frac{J_{13} R^{13} z}{1024 r^{26}} (1300075 z^{12} - 4056234 z^{10} r^2 + 4849845 z^8 r^4 - 2771340 z^6 r^6 \\ & \quad \left. + 765765 z^4 r^8 - 90090 z^2 r^{10} + 3003 r^{12}) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{J_{14} R^{14}}{2048 r^{28}} (5014575 z^{14} - 16900975 z^{12} r^2 + 22309287 z^{10} r^4 - 14549535 z^8 r^6 \\
& \quad + 4849845 z^6 r^8 - 765765 z^4 r^{10} + 45045 z^2 r^{12} - 429 r^{14}) \\
& + \frac{J_{15} R^{15} z}{2048 r^{30}} (9694845 z^{14} - 35102025 z^{12} r^2 + 50702925 z^{10} r^4 - 37182145 z^8 r^6 \\
& \quad + 14549535 z^6 r^8 - 2909907 z^4 r^{10} + 255255 z^2 r^{12} - 6435 r^{14}) \\
& + \frac{J_{16} R^{16}}{32768 r^{32}} (300540195 z^{16} - 1163381400 z^{14} r^2 + 1825305300 z^{12} r^4 - 1487285800 z^{10} r^6 \\
& \quad + 669278610 z^8 r^8 - 162954792 z^6 r^{10} + 19399380 z^4 r^{12} - 875160 z^2 r^{14} + 6435 r^{16}) \\
& + \frac{J_{17} R^{17} z}{32768 r^{34}} (583401555 z^{16} - 2404321560 z^{14} r^2 + 4071834900 z^{12} r^4 - 3650610600 z^{10} r^6 \\
& \quad + 1859107250 z^8 r^8 - 535422888 z^6 r^{10} + 81477396 z^4 r^{12} - 5542680 z^2 r^{14} + 109395 r^{16}) \\
& + \frac{J_{18} R^{18}}{65536 r^{36}} (2268783825 z^{18} - 9917826435 z^{16} r^2 + 18032411700 z^{14} r^4 - 17644617900 z^{12} r^6 \\
& \quad + 10039179150 z^{10} r^8 - 3346393050 z^8 r^{10} + 624660036 z^6 r^{12} - 58198140 z^4 r^{14} \\
& \quad + 2078505 z^2 r^{16} - 12155 r^{18}) \\
& + \frac{J_{19} R^{19} z}{65536 r^{38}} (4418157975 z^{18} - 20419054425 z^{16} r^2 + 39671305740 z^{14} r^4 - 42075627300 z^{12} r^6 \\
& \quad + 26466926850 z^{10} r^8 - 10039179150 z^8 r^{10} + 2230928700 z^6 r^{12} - 267711444 z^4 r^{14} \\
& \quad + 14549535 z^2 r^{16} - 230945 r^{18})
\end{aligned}$$

$$+ \frac{J_{20} R^{20}}{262144 r^{40}} (34461632205 z^{20} - 167890003050 z^{18} r^2 + 347123925225 z^{16} r^4 - 396713057400 z^{14} r^6 \\ + 273491577450 z^{12} r^8 - 116454478140 z^{10} r^{10} + 30117537450 z^8 r^{12} \\ - 4461857400 z^6 r^{14} + 334639305 z^4 r^{16} - 9699690 z^2 r^{18} + 46189 r^{20})$$

$$+ \frac{J_{21} R^{21} z}{262144 r^{42}} (67282234305 z^{20} - 344616322050 z^{18} r^2 + 755505013725 z^{16} r^4 - 925663800600 z^{14} r^6 \\ + 694247850450 z^{12} r^8 - 328189892940 z^{10} r^{10} + 97045398450 z^8 r^{12} \\ - 17210021400 z^6 r^{14} + 1673196525 z^4 r^{16} - 74364290 z^2 r^{18} + 969969 r^{20})$$

$$+ \frac{J_{22} R^{22}}{524288 r^{44}} (263012370465 z^{22} - 1412926920405 z^{20} r^2 + 3273855059475 z^{18} r^4 \\ - 4281195077775 z^{16} r^6 + 3471239252250 z^{14} r^8 - 1805044411170 z^{12} r^{10} \\ + 601681470390 z^{10} r^{12} - 124772655150 z^8 r^{14} + 15058768725 z^6 r^{16} \\ - 929553625 z^4 r^{18} + 22309287 z^2 r^{20} - 88179 r^{22})$$

$$+ \frac{J_{23} R^{23} z}{524288 r^{46}} (514589420475 z^{22} - 2893136075115 z^{20} r^2 + 7064634602025 z^{18} r^4 \\ - 9821565178425 z^{16} r^6 + 8562390155550 z^{14} r^8 - 4859734953150 z^{12} r^{10} \\ + 1805044411170 z^{10} r^{12} - 429772478850 z^8 r^{14} + 62386327575 z^6 r^{16} \\ - 5019589575 z^4 r^{18} + 185910725 z^2 r^{20} - 2028117 r^{22})$$

$$+ \frac{J_{24} R^{24}}{4194304 r^{48}} (8061900920775 z^{24} - 47342226683700 z^{22} r^2 + 121511715154830 z^{20} r^4 \\ - 178970743251300 z^{18} r^6 + 166966608033225 z^{16} r^8 - 102748681866600 z^{14} r^{10}$$

$$\begin{aligned}
& + 42117702927300 z^{12} r^{12} - 11345993441640 z^{10} r^{14} + 1933976154825 z^8 r^{16} \\
& - 194090796900 z^6 r^{18} + 10039179150 z^4 r^{20} - 202811700 z^2 r^{22} + 676039 r^{24})
\end{aligned}$$

$$\begin{aligned}
& + \frac{J_{25} R^{25} z}{4194304 r^{50}} (15801325804719 z^{24} - 96742811049300 z^{22} r^2 + 260382246760350 z^{20} r^4 \\
& - 405039050516100 z^{18} r^6 + 402684172315425 z^{16} r^8 - 267146572853160 z^{14} r^{10} \\
& + 119873462177700 z^{12} r^{12} - 36100888223400 z^{10} r^{14} + 7091245901025 z^8 r^{16} \\
& - 859544957700 z^6 r^{18} + 58227239070 z^4 r^{20} - 1825305300 z^2 r^{22} + 16900975 r^{24})
\end{aligned}$$

$$\frac{\partial F}{\partial x} = - \frac{\mu x}{r^3} \left[\frac{7 J_6 R^6}{16 r^{12}} (429 z^6 - 495 z^4 r^2 + 135 z^2 r^4 - 5 r^6) \right.$$

$$+ \frac{9 J_7 R^7 z}{16 r^{14}} (715 z^6 - 1001 z^4 r^2 + 385 z^2 r^4 - 35 r^6)$$

$$+ \frac{45 J_8 R^8}{128 r^{16}} (2431 z^8 - 4004 z^6 r^2 + 2002 z^4 r^4 - 308 z^2 r^6 + 7 r^8)$$

$$+ \frac{55 J_9 R^9 z}{128 r^{18}} (4199 z^8 - 7956 z^6 r^2 + 4914 z^4 r^4 - 1092 z^2 r^6 + 63 r^8)$$

$$+ \frac{33 J_{10} R^{10}}{256 r^{20}} (29393 z^{10} - 62985 z^8 r^2 + 46410 z^6 r^4 - 13650 z^4 r^6 + 1365 z^2 r^8 - 21 r^{10})$$

$$+ \frac{39 J_{11} R^{11} z}{256 r^{22}} (52003 z^{10} - 124355 z^8 r^2 + 106590 z^6 r^4 - 39270 z^4 r^6 + 5775 z^2 r^8 - 231 r^{10})$$

$$+ \frac{91 J_{12} R^{12}}{1024 r^{24}} (185725 z^{12} - 490314 z^{10} r^2 + 479655 z^8 r^4 - 213180 z^6 r^6 + 42075 z^4 r^8$$

$$- 2970 z^2 r^{10} + 33 r^{12})$$

$$+ \frac{105 J_{13} R^{13} z}{1024 r^{26}} (334305 z^{12} - 965770 z^{10} r^2 + 1062347 z^8 r^4 - 554268 z^6 r^6 + 138567 z^4 r^8 \\ - 14586 z^2 r^{10} + 429 r^{12})$$

$$+ \frac{15 J_{14} R^{14}}{2048 r^{28}} (9694845 z^{14} - 30421755 z^{12} r^2 + 37182145 z^{10} r^4 - 22309287 z^8 r^6 + 6789783 z^6 r^8 \\ - 969969 z^4 r^{10} + 51051 z^2 r^{12} - 429 r^{14})$$

$$+ \frac{17 J_{15} R^{15} z}{2048 r^{30}} (17678835 z^{14} - 59879925 z^{12} r^2 + 80528175 z^{10} r^4 - 54679625 z^8 r^6 + 19684665 z^6 r^8 \\ - 3594591 z^4 r^{10} + 285285 z^2 r^{12} - 6435 r^{14})$$

$$+ \frac{153 J_{16} R^{16}}{32768 r^{32}} (64822395 z^{16} - 235717800 z^{14} r^2 + 345972900 z^{12} r^4 - 262462200 z^{10} r^6 \\ + 109359250 z^8 r^8 - 24496472 z^6 r^{10} + 2662660 z^4 r^{12} - 108680 z^2 r^{14} + 715 r^{16})$$

$$+ \frac{171 J_{17} R^{17} z}{32768 r^{34}} (119409675 z^{16} - 463991880 z^{14} r^2 + 738168900 z^{12} r^4 - 619109400 z^{10} r^6 \\ + 293543250 z^8 r^8 - 78278200 z^6 r^{10} + 10958948 z^4 r^{12} - 680680 z^2 r^{14} + 12155 r^{16})$$

$$+ \frac{95 J_{18} R^{18}}{65536 r^{36}} (883631595 z^{18} - 3653936055 z^{16} r^2 + 6263890380 z^{14} r^4 - 5757717420 z^{12} r^6 \\ + 3064591530 z^{10} r^8 - 951080130 z^8 r^{10} + 164384220 z^6 r^{12} \\ - 14090076 z^4 r^{14} + 459459 z^2 r^{16} - 2431 r^{18})$$

$$+ \frac{105 J_{19} R^{19} z}{65536 r^{38}} (1641030105 z^{18} - 7195285845 z^{16} r^2 + 13223768580 z^{14} r^4 - 13223768580 z^{12} r^6 \\ + 7814045070 z^{10} r^8 - 2772725670 z^8 r^{10} + 573667380 z^6 r^{12} \\ - 63740820 z^4 r^{14} + 3187041 z^2 r^{16} - 46189 r^{18})$$

$$+ \frac{231 J_{20} R^{20}}{262144 r^{40}} (6116566755 z^{20} - 28345065450 z^{18} r^2 + 55599936075 z^{16} r^4 - 60108039000 z^{14} r^6 \\ + 39070225350 z^{12} r^8 - 15628090140 z^{10} r^{10} + 3780989550 z^8 r^{12} \\ - 521515800 z^6 r^{14} + 36216375 z^4 r^{16} - 965770 z^2 r^{18} + 4199 r^{20})$$

$$+ \frac{253 J_{21} R^{21} z}{262144 r^{42}} (11435320455 z^{20} - 55846913850 z^{18} r^2 + 116461247175 z^{16} r^4 - 135373757400 z^{14} r^6 \\ + 96042192750 z^{12} r^8 - 42807377340 z^{10} r^{10} + 11890938150 z^8 r^{12} \\ - 1972690200 z^6 r^{14} + 178562475 z^4 r^{16} - 7348250 z^2 r^{18} + 88179 r^{20})$$

$$+ \frac{69 J_{22} R^{22}}{524288 r^{44}} (171529806825 z^{22} - 880519675035 z^{20} r^2 + 1945334165775 z^{18} r^4 \\ - 2419805913525 z^{16} r^6 + 1861389164250 z^{14} r^8 - 915602237550 z^{12} r^{10} \\ + 287760703230 z^{10} r^{12} - 56057279850 z^8 r^{14} + 6329047725 z^6 r^{16} \\ - 363738375 z^4 r^{18} + 8083075 z^2 r^{20} - 29393 r^{22})$$

$$+ \frac{75 J_{23} R^{23} z}{524288 r^{46}} (322476036831 z^{22} - 1735881645069 z^{20} r^2 + 4050390505161 z^{18} r^4 \\ - 5369122297539 z^{16} r^6 + 4452442880886 z^{14} r^8 - 2397469243554 z^{12} r^{10}$$

$$+ 842354058546 z^{10} r^{12} - 189099890694 z^8 r^{14} + 25786348731 z^6 r^{16}$$

$$- 1940907969 z^4 r^{18} + 66927861 z^2 r^{20} - 676039 r^{22})$$

$$+ \frac{325 J_{24} R^{24}}{4194304 r^{48}} (1215486600363 z^{24} - 6846414320412 z^{22} r^2 + 16824699021438 z^{20} r^4$$

$$- 23679206030172 z^{18} r^6 + 21063479782653 z^{16} r^8 - 12329841823992 z^{14} r^{10}$$

$$+ 4794938487108 z^{12} r^{12} - 1221876216792 z^{10} r^{14} + 196372963413 z^8 r^{16}$$

$$- 18513276012 z^6 r^{18} + 895803678 z^4 r^{20} - 16848972 z^2 r^{22} + 52003 r^{24})$$

$$+ \frac{351 J_{25} R^{25} z}{4194304 r^{50}} (2295919134019 z^{24} - 13505406670700 z^{22} r^2 + 34865998853950 z^{20} r^4$$

$$- 51928083399500 z^{18} r^6 + 49331679229525 z^{16} r^8 - 31205155233560 z^{14} r^{10}$$

$$+ 13319273575300 z^{12} r^{12} - 3805506735800 z^{10} r^{14} + 707104292125 z^8 r^{16}$$

$$- 80811919100 z^6 r^{18} + 5142576670 z^4 r^{20} - 150808700 z^2 r^{22} + 1300075 r^{24}) \Big]$$

$$\frac{\partial F}{\partial y} = \frac{y}{x} \frac{\partial F}{\partial x}$$

$$\frac{\partial F}{\partial z} = - \frac{\mu}{r^3} \left[\frac{7 J_6 R^6 z}{16 r^{12}} (429 z^6 - 693 z^4 r^2 + 315 z^2 r^4 - 35 r^6)$$

$$+ \frac{J_7 R^7}{16 r^{14}} (6435 z^8 - 12012 z^6 r^2 + 6930 z^4 r^4 - 1260 z^2 r^6 + 35 r^8)$$

$$+ \frac{9 J_8 R^8 z}{128 r^{16}} (12155 z^8 - 25740 z^6 r^2 + 18018 z^4 r^4 - 4620 z^2 r^6 + 315 r^8)$$

$$+ \frac{5 J_9 R^9}{128 r^{18}} (46189 z^{10} - 109395 z^8 r^2 + 90090 z^6 r^4 - 30030 z^4 r^6 + 3465 z^2 r^8 - 63 r^{10})$$

$$+ \frac{11 J_{10} R^{10} z}{256 r^{20}} (88179 z^{10} - 230945 z^8 r^2 + 218790 z^6 r^4 - 90090 z^4 r^6 \\ + 15015 z^2 r^8 - 693 r^{10})$$

$$+ \frac{3 J_{11} R^{11}}{256 r^{22}} (676039 z^{12} - 1939938 z^{10} r^2 + 2078505 z^8 r^4 - 1021020 z^6 r^6 \\ + 225225 z^4 r^8 - 18018 z^2 r^{10} + 231 r^{12})$$

$$+ \frac{13 J_{12} R^{12} z}{1024 r^{24}} (1300075 z^{12} - 4056234 z^{10} r^2 + 4849845 z^8 r^4 - 2771340 z^6 r^6 \\ + 765765 z^4 r^8 - 90090 z^2 r^{10} + 3003 r^{12})$$

$$+ \frac{7 J_{13} R^{13}}{1024 r^{26}} (5014575 z^{14} - 16900975 z^{12} r^2 + 22309287 z^{10} r^4 - 14549535 z^8 r^6 + 4849845 z^6 r^8 \\ - 765765 z^4 r^{10} + 45045 z^2 r^{12} - 429 r^{14})$$

$$+ \frac{15 J_{14} R^{14} z}{2048 r^{28}} (9694845 z^{14} - 35102025 z^{12} r^2 + 50702925 z^{10} r^4 - 37182145 z^8 r^6 + 14549535 z^6 r^8 \\ - 2909907 z^4 r^{10} + 255255 z^2 r^{12} - 6435 r^{14})$$

$$+ \frac{J_{15} R^{15}}{2048 r^{30}} (300540195 z^{16} - 1163381400 z^{14} r^2 + 1825305300 z^{12} r^4 - 1487285800 z^{10} r^6 \\ + 669278610 z^8 r^8 - 162954792 z^6 r^{10} + 19399380 z^4 r^{12} - 875160 z^2 r^{14} + 6435 r^{16})$$

$$+ \frac{17 J_{16} R^{16} z}{32768 r^{32}} (583401555 z^{16} - 2404321560 z^{14} r^2 + 4071834900 z^{12} r^4 - 3650610600 z^{10} r^6$$

$$+ 1859107250 z^8 r^8 - 535422888 z^6 r^{10} + 81477396 z^4 r^{12}$$

$$- 5542680 z^2 r^{14} + 109395 r^{16})$$

$$+ \frac{9 J_{17} R^{17}}{32768 r^{34}} (2268783825 z^{18} - 9917826435 z^{16} r^2 + 18032411700 z^{14} r^4 - 17644617900 z^{12} r^6$$

$$+ 10039179150 z^{10} r^8 - 3346393050 z^8 r^{10} + 624660036 z^6 r^{12} - 58198140 z^4 r^{14}$$

$$+ 2078505 z^2 r^{16} - 12155 r^{18})$$

$$+ \frac{19 J_{18} R^{18} z}{65536 r^{36}} (4418157975 z^{18} - 20419054425 z^{16} r^2 + 39671305740 z^{14} r^4 - 42075627300 z^{12} r^6$$

$$+ 26466926850 z^{10} r^8 - 10039179150 z^8 r^{10} + 2230928700 z^6 r^{12} - 267711444 z^4 r^{14}$$

$$+ 14549535 z^2 r^{16} - 230945 r^{18})$$

$$+ \frac{5 J_{19} R^{19}}{65536 r^{38}} (34461632205 z^{20} - 167890003050 z^{18} r^2 + 347123925225 z^{16} r^4$$

$$- 396713057400 z^{14} r^6 + 273491577450 z^{12} r^8 - 116454478140 z^{10} r^{10} + 30117537450 z^8 r^{12}$$

$$- 4461857400 z^6 r^{14} + 334639305 z^4 r^{16} - 9699690 z^2 r^{18} + 46189 r^{20})$$

$$+ \frac{21 J_{20} R^{20} z}{262144 r^{40}} (67282234305 z^{20} - 344616322050 z^{18} r^2 + 755505013725 z^{16} r^4 - 925663800600 z^{14} r^6$$

$$+ 694247850450 z^{12} r^8 - 328189892940 z^{10} r^{10} + 97045398450 z^8 r^{12}$$

$$- 17210021400 z^6 r^{14} + 1673196525 z^4 r^{16} - 74364290 z^2 r^{18} + 969969 r^{20})$$

$$\begin{aligned}
& + \frac{11 J_{21} R^{21}}{262144 r^{42}} (263012370465 z^{22} - 1412926920405 z^{20} r^2 + 3273855059475 z^{18} r^4 \\
& - 4281195077775 z^{16} r^6 + 3471239252250 z^{14} r^8 - 1805044411170 z^{12} r^{10} \\
& + 601681470390 z^{10} r^{12} - 124772655150 z^8 r^{14} + 15058768725 z^6 r^{16} \\
& - 929553625 z^4 r^{18} + 22309287 z^2 r^{20} - 88179 r^{22})
\end{aligned}$$

$$\begin{aligned}
& + \frac{23 J_{22} R^{22} z}{524288 r^{44}} (514589420475 z^{22} - 2893136075115 z^{20} r^2 + 7064634602025 z^{18} r^4 \\
& - 9821565178425 z^{16} r^6 + 8562390155550 z^{14} r^8 - 4859734953150 z^{12} r^{10} \\
& + 1805044411170 z^{10} r^{12} - 429772478850 z^8 r^{14} + 62386327575 z^6 r^{16} \\
& - 5019589575 z^4 r^{18} + 185910725 z^2 r^{20} - 2028117 r^{22})
\end{aligned}$$

$$\begin{aligned}
& + \frac{3 J_{23} R^{23}}{524288 r^{46}} (8061900920775 z^{24} - 47342226683700 z^{22} r^2 + 121511715154830 z^{20} r^4 \\
& - 178970743251300 z^{18} r^6 + 166966608033225 z^{16} r^8 - 102748681866600 z^{14} r^{10} \\
& + 42117702927300 z^{12} r^{12} - 11345993441640 z^{10} r^{14} + 1933976154825 z^8 r^{16} \\
& - 194090796900 z^6 r^{18} + 10039179150 z^4 r^{20} - 202811700 z^2 r^{22} + 676039 r^{24})
\end{aligned}$$

$$\begin{aligned}
& + \frac{25 J_{24} R^{24} z}{4194304 r^{48}} (15801325804719 z^{24} - 96742811049300 z^{22} r^2 + 260382246760350 z^{20} r^4 \\
& - 405039050516100 z^{18} r^6 + 402684172315425 z^{16} r^8 - 267146572853160 z^{14} r^{10} \\
& + 119873462177700 z^{12} r^{12} - 36100888223400 z^{10} r^{14} + 7091245901025 z^8 r^{16} \\
& - 859544957700 z^6 r^{18} + 58227239070 z^4 r^{20} - 1825305300 z^2 r^{22} + 16900975 r^{24})
\end{aligned}$$

$$\begin{aligned}
& + \frac{13 J_{25} R^{25}}{4194304 r^{50}} (61989816618513 z^{26} - 395033145117975 z^{24} r^2 + 1112542327066950 z^{22} r^4 \\
& - 1822675727322450 z^{20} r^6 + 1923935489951475 z^{18} r^8 - 1369126185872445 z^{16} r^{10} \\
& + 667866432132900 z^{14} r^{12} - 222622144044300 z^{12} r^{14} + 49638721307175 z^{10} r^{16} \\
& - 7091245901025 z^8 r^{18} + 601681470390 z^6 r^{20} - 26466926850 z^4 r^{22} \\
& + 456326325 z^2 r^{24} - 1300075 r^{26}) \Big]
\end{aligned}$$